

Integer powers of certain complex tridiagonal matrices and some complex factorizations

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Abstract

In this paper, we obtain a general expression for the entries of the r th ($r \in \mathbb{Z}$) power of a certain $n \times n$ complex tridiagonal matrix. In addition, we get the complex factorizations of Fibonacci polynomials, Fibonacci and Pell numbers.

1 Introduction

In order to solve some difference equations, differential and delay differential equations and boundary value problems, we need to compute the arbitrary integer powers of a square matrix.

The integer powers of an $n \times n$ matrix A is computed using the well-known expression $A^r = PJ^rP^{-1}$ [?], where J is the Jordan's form and P is the eigenvector matrix of A , respectively.

Recently, the calculations of integer powers and eigenvalues of tridiagonal matrices have been well studied. For instance, Rimas [1-4] obtained the positive integer powers of certain tridiagonal matrices of odd and even order. Öteleş and Akbulak [6,7] generalized the results obtained in [1-4]. Gutiérrez [8,10] calculated the powers of tridiagonal matrices with constant diagonal. For detailed information on the powers and the eigenvalues of tridiagonal matrices, we may refer to the reader [5,9].

In [12], Cahill et al. considered the following tridagonal matrix

$$H(n) = \begin{pmatrix} h_{1,1} & h_{1,2} & & & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} & & \\ & h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & h_{n-1,n} \\ 0 & & & h_{n,n-1} & h_{n,n} \end{pmatrix}$$

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and using the successive determinants they computed determinant of $H(n)$ as

$$|H(n)| = h_{n,n} |H(n-1)| - h_{n-1,n} h_{n,n-1} |H(n-2)|$$

with initial conditions $|H(1)| = h_{1,1}$, $|H(2)| = h_{1,1}h_{2,2} - h_{1,2}h_{2,1}$.

Let $\{H^\dagger(n), n = 1, 2, \dots\}$ be the sequence of tridiagonal matrices as in the form

$$H^\dagger(n) = \begin{pmatrix} h_{1,1} & -h_{1,2} & & & \\ -h_{2,1} & h_{2,2} & -h_{2,3} & & \\ & -h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & -h_{n-1,n} \\ & & & -h_{n,n-1} & h_{n,n} \end{pmatrix}.$$

Then

$$\det(H(n)) = \det(H^\dagger(n)). \quad (1)$$

Let T and T^\dagger be $n \times n$ tridiagonal matrices as the following

$$T := \begin{pmatrix} 0 & 2 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 0 & 1 \\ & & & & 2 & 0 \end{pmatrix} [1],$$

$$T^\dagger := \begin{pmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{pmatrix} [2].$$

From (1), it is clear that

$$|\lambda I_n - T| = \begin{vmatrix} \mu & -2 & & & \\ -1 & \mu & -1 & & \\ & -1 & \mu & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & \mu & -1 \\ & & & & -2 & \mu \end{vmatrix}$$

and

$$|\lambda I_n - T^\dagger| = \begin{vmatrix} \mu^\dagger - 1 & -1 & & & \\ -1 & \mu^\dagger & -1 & & \\ & -1 & \mu^\dagger & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & \mu^\dagger & -1 \\ & & & & -1 & \mu^\dagger - 1 \end{vmatrix}.$$

By [1, p. 3] and [2, p. 2], the eigenvalues of T and T^\dagger are obtained as

$$\mu_k = 2 \cos \left(\frac{(k-1)\pi}{n-1} \right), \quad k = \overline{1, n} \quad [1, p.3]$$

and

$$\mu_k^\dagger = -2 \cos \left(\frac{k\pi}{n} \right), \quad k = \overline{1, n} \quad [2, p.2]$$

respectively.

Let

$$A := \begin{pmatrix} a & 2b & & & 0 \\ b & a & -b & & \\ & -b & a & -b & \\ & & \ddots & \ddots & \ddots \\ 0 & & & -b & a & b \\ & & & & 2b & a \end{pmatrix} \quad (2)$$

and

$$A^\dagger := \begin{pmatrix} a+b & b & & & 0 \\ b & a & -b & & \\ & -b & a & -b & \\ & & \ddots & \ddots & \ddots \\ 0 & & & -b & a & b \\ & & & & b & a+b \end{pmatrix} \quad (3)$$

be the tridiagonal matrices with a and $b \neq 0$ are complex numbers. In this paper, we obtain the eigenvalues and eigenvectors of an $n \times n$ complex tridiagonal matrices in (2) and (3) and calculate the integer powers of the matrix in (2) for n is odd order.

2 Eigenvalues and eigenvectors of A and A^\dagger

Theorem 1 *Let A be an $n \times n$ tridiagonal matrix given by (2). Then the eigenvalues and eigenvectors of A are*

$$\lambda_k = a + 2b \cos \left(\frac{(k-1)\pi}{n-1} \right), \quad k = \overline{1, n} \quad (4)$$

and

$$x_{jk} = \begin{cases} T_{j-1}(m_k), & j = 1, 2, n-1, n \\ (-1)^j T_{j-1}(m_k), & j = \overline{3, n-2} \end{cases}; k = \overline{1, n}$$

where $m_k = \frac{\lambda_k - a}{2b}$, $T_s(\cdot)$ is the s -th degree Chebyshev polynomial of the first kind [11, p. 14].

Proof. Let B be the following $n \times n$ tridiagonal matrix

$$B := \begin{pmatrix} c & 2 & & & \\ 1 & c & -1 & & \\ & -1 & c & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & c & 1 \\ & & & & 2 & c \end{pmatrix} \quad (5)$$

where $c = \frac{a}{b}$. Then the characteristic polynomials of B are

$$p_n(t) = (t^2 - 4)\Delta_{n-2}(t) \quad (6)$$

where $t = \lambda - c$ and

$$\Delta_n(t) = t\Delta_{n-1}(t) - \Delta_{n-2}(t) \quad (7)$$

with initial conditions $\Delta_0(t) = 1$, $\Delta_1(t) = t$ and $\Delta_2(t) = t^2 - 1$. Note that the solution of the difference equation in (7) is $\Delta_n(t) = U_n(\frac{t}{2})$, where U_n is the n th degree Chebyshev polynomial of the second kind [11, p.15]. i.e.

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}, \quad -1 \leq x \leq 1.$$

All the roots of the polynomial $U_n(x)$ are included in the interval $[-1, 1]$ and can be found using the relation

$$x_{nk} = \cos\left(\frac{k\pi}{n+1}\right), \quad k = \overline{1, n}.$$

Therefore the characteristic polynomial in (6) can be written as

$$p_n(t) = (t^2 - 4)U_{n-2}\left(\frac{t}{2}\right).$$

From [6, p. 2], the eigenvalues of the matrix B are

$$t_k = 2 \cos\left(\frac{(k-1)\pi}{n-1}\right), \quad k = \overline{1, n}.$$

Then we get the eigenvalues of the matrix A as

$$\lambda_k = a + 2b \cos\left(\frac{(k-1)\pi}{n-1}\right).$$

Now we compute eigenvectors of the matrix A . All eigenvectors of the matrix A are the solutions of the following homogeneous linear equations system

$$(\lambda_k I_n - A)x = 0 \quad (8)$$

where λ_k is the k th eigenvalue of the matrix A ($k = \overline{1, n}$). The equations system (8) is clearly written as

$$\left. \begin{aligned} (\lambda_k - a)x_1 - 2bx_2 &= 0 \\ -bx_1 + (\lambda_k - a)x_2 + bx_3 &= 0 \\ bx_2 + (\lambda_k - a)x_3 + bx_4 &= 0 \\ &\vdots \\ bx_{n-2} + (\lambda_k - a)x_{n-1} - bx_n &= 0 \\ -2bx_{n-1} + (\lambda_k - a)x_n &= 0 \end{aligned} \right\} \quad (9)$$

Dividing all terms of the each equation in system (9) by $b \neq 0$, substituting $m_k = \frac{\lambda_k - a}{2b}$, choosing $x_1 = 1$ then solving the set of the system (9) we find the j th component of k th eigenvector of the matrix A as

$$x_{jk} = \begin{cases} T_{j-1}(m_k), & j = 1, 2, n-1, n \\ (-1)^j T_{j-1}(m_k), & j = 3, n-2 \end{cases} ; j, k = \overline{1, n} \quad (10)$$

where $m_k = \frac{\lambda_k - a}{2b}$ and $T_s(\cdot)$ is the s -th degree Chebyshev polynomial of the first kind. ■

Theorem 2 *Let A^\dagger be an $n \times n$ tridiagonal matrix given by (3). Then the eigenvalues and eigenvectors of A^\dagger are*

$$\lambda_k^\dagger = a - 2b \cos\left(\frac{k\pi}{n}\right), \quad k = \overline{1, n}$$

and

$$y_{jk}^\dagger = \begin{cases} T_{\frac{2j-1}{2}}(m_k^\dagger), & j = 1, 2, n-1, n \\ (-1)^j T_{\frac{2j-1}{2}}(m_k^\dagger), & j = 3, n-2 \end{cases} ; k = \overline{1, n}$$

where $m_k^\dagger = \frac{\lambda_k^\dagger - a}{2b}$ and $T_s(\cdot)$ is the s -th degree Chebyshev polynomial of the first kind [11, p. 14].

Proof. Let

$$S := \begin{pmatrix} \frac{a}{b} + 1 & 1 & & & 0 \\ 1 & \frac{a}{b} & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \frac{a}{b} & -1 \\ 0 & & & -1 & \frac{a}{b} & 1 \\ & & & & 1 & \frac{a}{b} + 1 \end{pmatrix}.$$

From [7, Lemma 2, p. 65], we have the eigenvalues of the matrix S as

$$\delta_k = \frac{a}{b} - 2 \cos\left(\frac{k\pi}{n}\right), \quad \text{for } k = \overline{1, n}$$

Since the eigenvalues of A^\dagger are $\lambda_k^\dagger = b\delta_k$, the proof is completed.

The eigenvectors of A^\dagger is the solution of the following linear homogeneous equations system:

$$(\lambda_j^\dagger I_n - A^\dagger)y_{jk} = 0 \quad (11)$$

where λ_j^\dagger and y_{jk} are the j th eigenvalues and k th eigenvectors of the matrix A^\dagger for $1 \leq j, k \leq n$. Then the solution of the equations system in (11) is

$$y_{jk} = \begin{cases} T_{\frac{2j-1}{2}}(m_k^\dagger), & j = 1, 2, n-1, n \\ (-1)^j T_{\frac{2j-1}{2}}(m_k^\dagger), & j = \overline{3, n-2} \end{cases}; k = \overline{1, n}$$

here $m_k^\dagger = \frac{\lambda_k^\dagger - a}{2b}$ and $T_s(\cdot)$ is the s -th degree Chebyshev polynomial of the first kind. ■

3 The integer powers of the matrix A

In this section we assume that n is positive odd integer.

Since all the eigenvalues λ_k ($k = \overline{1, n}$) are simple, each eigenvalue λ_k corresponds single Jordan cells $J_1(\lambda_k)$ in the matrix J . Then, we write down the Jordan's forms of the matrix A

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n).$$

Using the equality $J = P^{-1}AP$, we need the matrices P and P^{-1} and derive the expressions for the r th power ($r \in \mathbb{Z}$) of the matrix A . Then

$$A^r = PJ^rP^{-1}.$$

From (10), we can write the eigenvectors matrix P as

$$P = [x_{jk}] = \begin{cases} T_{j-1}(m_k), & j = 1, 2, n-1, n \\ (-1)^j T_{j-1}(m_k), & j = \overline{3, n-2} \end{cases} \quad k = \overline{1, n}$$

where $T_s(\cdot)$ is the s -th degree Chebyshev polynomial of the first kind.

First of all, let us obtain the inverse matrix P^{-1} .

Denoting j th column of the matrix P^{-1} by p_j then we have

$$p_j = \begin{pmatrix} 2T_{j-1}(m_1) \\ T_{j-1}(m_2) \\ T_{j-1}(m_3) \\ 2T_{j-1}(m_4) \\ \vdots \\ 2T_{j-1}(m_n) \end{pmatrix}, \quad j = 1, n$$

and

$$p_j = \begin{pmatrix} (-1)^j 4T_{j-1}(m_1) \\ (-1)^j 2T_{j-1}(m_2) \\ (-1)^j 2T_{j-1}(m_3) \\ (-1)^j 4T_{j-1}(m_4) \\ \vdots \\ (-1)^j 4T_{j-1}(m_n) \end{pmatrix}, \quad j = \overline{2, n-1}.$$

Hence we obtain

$$P^{-1} = \frac{1}{2n-2}(p_1, p_2, \dots, p_n).$$

Let

$$A^r = PJ^rP^{-1} = U(r) = (u_{ij}(r)).$$

Then

$$PJ^r = \begin{cases} \lambda_k^r T_{j-1}(m_k), & j = 1, 2, n-1, n \\ (-1)^j \lambda_k^r T_{j-1}(m_k), & j = \overline{3, n-2} \end{cases} \quad k = \overline{1, n}.$$

Hence

$$\begin{aligned} u_{ij}(r) &= \frac{1}{2n-2} (\lambda_2^r T_{i-1}(m_2) T_{j-1}(m_2) + \lambda_3^r T_{i-1}(m_3) T_{j-1}(m_3)) \\ &\quad + 2 \sum_{\substack{k=1 \\ k \neq 2, 3}}^n \lambda_k^r T_{i-1}(m_k) T_{j-1}(m_k) \end{aligned}$$

where $i = \overline{1, n}$; $j = 1, n$ and

$$\begin{aligned} u_{ij}(r) &= \frac{1}{n-1} \left((-1)^j (\lambda_2^r T_{i-1}(m_2) T_{j-1}(m_2) + \lambda_3^r T_{i-1}(m_3) T_{j-1}(m_3)) \right. \\ &\quad \left. + (-1)^j 2 \sum_{\substack{k=1 \\ k \neq 2, 3}}^n \lambda_k^r T_{i-1}(m_k) T_{j-1}(m_k) \right) \end{aligned}$$

here $i = \overline{1, n}$; $j = \overline{2, n-1}$.

4 Numerical examples

We can find the arbitrary integer powers of the n th order of the matrix, where n is positive odd integer.

Example 3 Let $n = 3, r = 3, a = 1$ and $b = 3$. Then we have

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3) = \text{diag}(a, a + 2b, a - 2b) = \text{diag}(1, 7, -5).$$

Therefore

$$A^3 = (q_{ij}(r)) = (q_{ij}(3)) = \frac{1}{4} \begin{pmatrix} 55 & 234 & 54 \\ 117 & 109 & 117 \\ 54 & 234 & 55 \end{pmatrix}.$$

Example 4 If $n = 5, r = 4, a = 1$ and $b = 3$, then

$$\begin{aligned} J &= \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \\ &= \text{diag}(1, 7, -5, 1 + 3\sqrt{2}, 1 - 3\sqrt{2}). \end{aligned}$$

Therefore

$$A^4 = q_{ij}(4) = \frac{1}{8} \begin{pmatrix} 595 & 672 & -756 & 216 & 162 \\ 336 & 973 & -444 & 540 & 108 \\ -378 & -444 & 757 & -444 & -378 \\ 108 & 540 & -444 & 973 & 336 \\ 162 & 216 & -756 & 672 & 595 \end{pmatrix}.$$

5 Complex Factorizations

The well-known $F(x) = \{F_n(x)\}_{n=1}^{\infty}$ Fibonacci polynomials are defined by $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$. For example if $x = 1$ and $x = 2$, then we obtain

$$F_n(1) = \{0, 1, 1, 2, 3, 5, 8, \dots\}$$

Fibonacci numbers and

$$F_n(2) = \{0, 1, 2, 5, 12, 29, \dots\}$$

Pell numbers, respectively.

Theorem 5 Let the matrix A be as in (2) with $a := x$ and $b := \mathbf{i}$ where $\mathbf{i} = \sqrt{-1}$. Then

$$\det(A) = (x^2 + 4)F_{n-1}(x).$$

Proof. Applying Laplace expansion according to the first two and the last two rows of the matrix A , we have

$$\det(A) = x^2 D_{n-2} + 4x D_{n-3} + 4 D_{n-4}$$

here $D_n = \det(\text{tridiag}_n(-\mathbf{i}, x, -\mathbf{i}))$. Since

$$\det(\text{tridiag}_n(-\mathbf{i}, x, -\mathbf{i})) = F_{n+1}(x),$$

we obtain

$$\begin{aligned} \det(A) &= x^2 F_{n-1}(x) + 4x F_{n-2}(x) + 4 F_{n-3}(x) \\ &= x^2 (x F_{n-2}(x) + F_{n-3}(x)) + 4x F_{n-2}(x) + 4 F_{n-3}(x) \\ &= (x^2 + 4)(x F_{n-2}(x) + F_{n-3}(x)) = (x^2 + 4) F_{n-1}(x). \end{aligned}$$

Thus, the proof is completed. ■

Corollary 6 *Let the matrix A be as in (2) with $a := x$ and $b := \mathbf{i}$. Then the complex factorization of generalized Fibonacci-Pell numbers is the following form:*

$$F_{n-1}(x) = \frac{1}{x^2 + 4} \prod_{k=1}^n \left(x + 2\mathbf{i} \cos \left(\frac{(k-1)\pi}{n-1} \right) \right)$$

Proof. Since the eigenvalues of the matrix A from (4)

$$\lambda_j = x + 2\mathbf{i} \cos \left(\frac{(j-1)\pi}{n-1} \right), \quad j = \overline{1, n},$$

the determinant of the matrix A can be obtained as

$$\det(A) = \prod_{k=1}^n \left(x + 2\mathbf{i} \cos \left(\frac{(k-1)\pi}{n-1} \right) \right). \quad (12)$$

By considering (12) and Theorem 5, the complex factorization of generalized Fibonacci-Pell numbers is obtained. ■

Theorem 7 *Let the matrix A^\dagger be as in (3). $a := 1$ and $b := \mathbf{i}$ where $\mathbf{i} = \sqrt{-1}$. Then*

$$\det(A^\dagger) = \begin{cases} (1 + 2\mathbf{i})F_n, & \text{if } a = 1 \text{ and } b = \mathbf{i} \\ (2 + 2\mathbf{i})P_n, & \text{if } a = 2 \text{ and } b = \mathbf{i} \end{cases}$$

where F_n and P_n are n th Fibonacci and Pell numbers, respectively.

Proof. Applying Laplace expansion according to the first two and last two rows the determinant of the matrix A^\dagger , we have

$$\begin{aligned} \det(A^\dagger) &= (a+b)^2 \det(\text{tridiag}_{n-2}(-b, a, -b)) \\ &\quad - 2b^2(a+b) \det(\text{tridiag}_{n-3}(-b, a, -b)) \\ &\quad + b^4 \det(\text{tridiag}_{n-4}(-b, a, -b)). \end{aligned} \quad (13)$$

If we get $a = 1$ and $b = \mathbf{i}$ in (13), then we obtain

$$\begin{aligned} \det(A^\dagger) &= (1 + \mathbf{i})^2 \det(\text{tridiag}_{n-2}(-\mathbf{i}, 1, -\mathbf{i})) \\ &\quad + 2(1 + \mathbf{i}) \det(\text{tridiag}_{n-3}(-\mathbf{i}, 1, -\mathbf{i})) \\ &\quad + \det(\text{tridiag}_{n-4}(-\mathbf{i}, 1, -\mathbf{i})). \end{aligned}$$

Since

$$\det(\text{tridiag}_n(\mathbf{i}, 1, \mathbf{i})) = \det(\text{tridiag}_n(-\mathbf{i}, 1, -\mathbf{i}))$$

from (1), we write

$$\begin{aligned} \det(A^\dagger) &= (1 + \mathbf{i})^2 F_{n-1} + 2(1 + \mathbf{i}) F_{n-2} + F_{n-3} \\ &= (1 + 2\mathbf{i}) F_n. \end{aligned}$$

Similarly, we can easily obtain Pell numbers. ■

Corollary 8 *Let the matrix A^\dagger be as in (3) with $a := 2$ and $b := \mathbf{i}$. Then the complex factorization of Fibonacci and Pell numbers are*

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2\mathbf{i} \cos \left(\frac{k\pi}{n} \right) \right)$$

and

$$P_n = \prod_{k=1}^{n-1} \left(2 - 2\mathbf{i} \cos \left(\frac{k\pi}{n} \right) \right).$$

Proof. Since the eigenvalues of the matrix A^\dagger are

$$\lambda_k = a - 2b \cos \cos \left(\frac{k\pi}{n} \right), \quad k = \overline{1, n}$$

and the determinant of the matrix A^\dagger is multiplication of its eigenvalues, we have

$$\begin{aligned} F_n &= \frac{1}{1 + 2\mathbf{i}} \prod_{k=1}^n \left(1 - 2\mathbf{i} \cos \left(\frac{k\pi}{n} \right) \right) \\ &= \prod_{k=1}^{n-1} \left(1 - 2\mathbf{i} \cos \left(\frac{k\pi}{n} \right) \right) \end{aligned}$$

and

$$\begin{aligned} P_n &= \frac{1}{2 + 2\mathbf{i}} \prod_{k=1}^n \left(2 - 2\mathbf{i} \cos \left(\frac{k\pi}{n} \right) \right) \\ &= \prod_{k=1}^{n-1} \left(2 - 2\mathbf{i} \cos \left(\frac{k\pi}{n} \right) \right). \end{aligned}$$

Thus, the proof is completed. ■

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